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Boundary conditions for plane flows of smooth, nearly elastic, circular disks

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We consider plane flows of identical, smooth, nearly elastic, circular disks interacting with a boundary formed by attaching halves of similar disks at equal intervals along a flat wall. The roughness of the boundary is given in terms of the diameters of the two types of disks and the spacing of the wall disks. We suppose that the velocity distribution of the flow disks is Maxwellian and calculate the rates at which momentum and energy are supplied to the flow disks in collisions over a unit length of the boundary. At the boundary we balance these supplies with the stress and the total flux of energy in the flow and obtain boundary conditions on the shear stress, pressure, and flux of fluctuation energy. We find that the boundary can either supply fluctuation energy to the flow or absorb it, depending on the relative magnitudes of the rate of working of the boundary tractions through the slip velocity and the rate at which energy is dissipated in collisions. As an example we solve the boundary-value problem for the steady shearing flow maintained by the relative motion of parallel plates a fixed distance apart. When the dimensions and properties of the flow disks and the boundary are given, the specification of the distance between the plates and their relative velocity determines the slip velocity, the shear stress and pressure necessary to maintain the flow, and the distributions of mean velocity, fluctuation energy, and density.

1. Introduction

In the past few years there has been a resurgence of interest in the study of rapid deformations of granular materials. Such deformations of relatively dense systems are important in many materials-handling applications in the mining, cereal, and pharmaceutical industries and often occur naturally in the form of avalanches, rock debris slides, and pack-ice flows.

Experiments on steady shearing flows of idealized materials, inspired by Bagnold's (1954) pioneering work, have been carried out by Savage (1978), Savage & McKeown (1983), Savage & Sayed (1984), and Hanes & Inman (1985) and form the experimental basis for the development of theory. In addition, the unique numerical simulations of the detailed dynamics of circular disks in simple plane flows by Campbell & Brennen (1983, 1985), Walton (1983), and Campbell & Gong (1986) provide extremely detailed information against which theory can be compared.

Recent theories for rapid deformations of granular materials depart from Bagnold's (1954) heuristic treatment of grain collisions and take into account explicitly the energy associated with the velocity fluctuations of the grains. These theories differ mainly in the degrees of detail employed when calculating averages.

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The most detailed development of theory employs methods from the kinetic theory of dense gases, extended to account for the energy dissipated in collisions, to determine the single-particle velocity distribution function as an approximate solution to its equation of evolution. The influence of the dense nature of the system on the frequency of collisions is taken into account in a way proposed by Enskog (Chapman & Cowling 1970). Lun *et al.* (1984) and Jenkins & Richman (1985*a*) provide such a theory for identical, smooth, nearly elastic spheres and Jenkins & Richman (1985*b*) do the same for identical, rough, inelastic, circular disks, at least when the energy dissipated in collisions is not too great.

In similar, but somewhat cruder approaches, the single-particle velocity distribution function is assumed to be Maxwellian and Enskog's treatment of collisions is employed. Theories of this type include Savage & Jeffrey's (1981) original introduction of methods from the kinetic theory into the subject, the extension of their results from identical, smooth, elastic spheres to nearly elastic spheres by Jenkins & Savage (1983), the derivation of stress relations for the homogeneous shearing of identical, rough, inelastic spheres by Lun & Savage (1986), and theory for binary mixtures of smooth, nearly elastic, circular disks or spheres by Jenkins & Mancini (1986).

Other simpler and perhaps more transparent methods of averaging have been employed by Ogawa, Umemura & Oshima (1980) and Ackermann & Shen (1983) for spheres, by Shen & Ackermann (1984) for circular disks, and by Shen (1984) for a binary mixture of spheres. A phenomenological theory proposed by Haff (1983) has the same physical foundations and essentially the same structure as the kinetic theories for dense systems of smooth, nearly elastic spheres or disks.

These theories have all been tested against both the experiments on shearing flow and the corresponding numerical simulations, typically by determining the predicted magnitudes of the shear stress and the pressure in simple shear. However this simple flow, involving a uniform velocity gradient, fluctuation energy, and density is probably exceptional, given that slip is inevitably observed at the boundaries in the experiments and that, in general, fluctuation energy will be removed from or supplied to the flow at the boundaries. What is required in order to make a fair comparison with the experiments, are derivations of boundary conditions that are as detailed as the derivations of the constitutive relations for the flow.

One such derivation has been provided by Hui *et al.* (1984) who operate in the spirit of Haff's (1983) phenomenological theory and characterize a boundary in terms of its coefficient of restitution in a collision with a flow particle and a roughness parameter. They derive expressions for the rates at which energy is dissipated and tangential momentum is transferred in collisions over a unit area of the boundary. They equate the former to the flux of fluctuation energy from the flow and the latter to the shear stress in the flow, and obtain boundary conditions relating the fluctuation energy to its derivative normal to the wall and the slip velocity to the normal derivative of the flow velocity. They employ these boundary conditions with the balance laws and constitutive relations of Haff's (1983) theory to analyse a steady shearing flow maintained between parallel flat plates by their relative motion.

Here we consider plane flows of identical, smooth, nearly elastic, circular disks interacting with a boundary composed of halves of similar disks attached at equal intervals to a flat wall. We first employ methods of averaging from the kinetic theory to derive expressions for the rate at which linear momentum and energy are supplied to the flow disks in collisions with the boundary. Then, at the boundary, we balance these supplies with the corresponding quantities in the flow. The resulting boundary conditions differ from those of Hui *et al.* (1984) in three major respects.

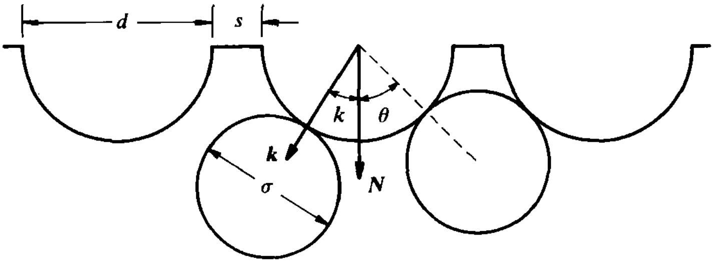

FIGURE 1. The geometry of the boundary and collisions.

First, an explicit measure of the roughness of the boundary is determined in terms of the diameters of the two types of disks and the spacing of the wall disks.

Secondly, the boundary condition on the flux of fluctuation energy contains the rate of working of the boundary tractions through the slip velocity in addition to the rate at which energy is dissipated in collisions. Consequently the boundary may either supply fluctuation energy to the flow or absorb it, depending upon the relative size of these two terms. Their relative size depends upon the diameters of the two types of disks, the energy lost in the two types of collisions, and the spacing of the wall disks. The exceptional case, in which the two terms balance, corresponds to a homogeneous shearing flow with uniform velocity gradient, fluctuation energy, and area fraction.

Thirdly, a boundary condition on the flow pressure is obtained. In our view, this condition fixes the density or, equivalently, the area fraction of the flow disks at a boundary and, for example, forces a unique solution to the boundary-value problem for steady shearing between parallel plates. That is, given the properties of the two types of disks, the spacing of the wall disks, the distance between the plates, and their relative velocity, there are unique distributions of mean velocity, fluctuation energy, and area fraction corresponding to uniquely determined values of the shear stress, pressure, and slip velocity at the plates.

In Appendix A we sketch a similar derivation of boundary conditions for identical, smooth, nearly elastic spheres interacting with a bumpy boundary. The boundary is assumed to be constructed by attaching identical, smooth, nearly elastic hemispheres to a flat wall. The centres of the hemispheres are assumed to be randomly distributed over the wall with a fixed mean spacing.

2. Boundary geometry

We consider a plane flow of identical, smooth, circular disks with mass m and diameter σ interacting with a bumpy boundary. The geometry of the boundary is shown in figure 1. Halves of identical, smooth, circular disks are equally spaced along a flat wall. The diameter of each wall disk is d and the spacing between them is s. The maximum spacing between the disks is fixed by the requirement that a flow disk never collides with the flat wall. Consequently, the range of the spacing is given in terms of the diameters of the two types of disks by $0 \leqslant s/d \leqslant -1+(1+2\sigma/d)^{\frac{1}{2}}$. The number α of wall disks per unit length of the wall is $1/(d+s)$.

Because of the presence of neighbouring half disks, only a fraction of the periphery of a wall disk is accessible to flow disks. From figure 1, this fraction is equal to $2\theta/\pi$, where $\sin\theta = (d+s)/(d+\sigma)$. As θ increases, more of the periphery of a wall disk is available for collisions and, consequently, the boundary appears rougher. The magnitude of $\sin\theta$ is a natural measure of this roughness.

3. Boundary collisions

We suppose that the wall is translating with velocity $\boldsymbol{U}$ and consider a collision between a flow disk and a wall disk. At collision, the orientation of the line of centres is given by the unit vector $\boldsymbol{k}$, directed from the centre of the wall disk to the centre of the flow disk. As indicated in figure 1, neighbouring wall disks will not prohibit the collision provided that the angle k between $\boldsymbol{k}$ and the unit inward normal $\boldsymbol{N}$ to the wall is between $-\theta$ and $+\theta$.

The velocity $\boldsymbol{c}$ of the centre of a flow disk immediately before the collision is related to its velocity $\boldsymbol{c}'$ immediately after the collision through

$$m\boldsymbol{c}' = m\boldsymbol{c}+\boldsymbol{J}, \tag{1}$$

where $\boldsymbol{J}$ is the impulse exerted by the wall disk on the flow disk. We assume that the velocity $\boldsymbol{U}$ of the wall disk is unaffected by the collision.

We characterize the energy lost in the collision by assuming that the relative velocity $\boldsymbol{g} \equiv \boldsymbol{U}-\boldsymbol{c}$ of the particle centres just before the collision is related to that, $\boldsymbol{g}' \equiv \boldsymbol{U}-\boldsymbol{c}'$, just after the collision through the coefficient of restitution e_{w},

$$(\boldsymbol{g}\cdot\boldsymbol{k}) = -e_{\mathrm{w}}(\boldsymbol{g}\cdot\boldsymbol{k}). \tag{2}$$

For smooth disks, (1) and (2) may be used to determine the impulse $\boldsymbol{J}$; then, upon eliminating $\boldsymbol{J}$ from (1), we may write the relation between the velocities before and after collision as

$$\boldsymbol{c}' = \boldsymbol{c}+(1+e_{\mathrm{w}})\,(\boldsymbol{g}\cdot\boldsymbol{k})\,\boldsymbol{k}. \tag{3}$$

From this it follows easily that

$$c'^2 = c^2+2(1+e_{\mathrm{w}})\,(\boldsymbol{g}\cdot\boldsymbol{k})\,(\boldsymbol{U}\cdot\boldsymbol{k})-(1-e_{\mathrm{w}}^2)\,(\boldsymbol{g}\cdot\boldsymbol{k})^2. \tag{4}$$

where, for example, $c^2 \equiv \boldsymbol{c}\cdot\boldsymbol{c}$.

4. Mean collisional rates of supply

We first determine an expression for the probable frequency of collisions per unit length of the flat wall.

At time t, the probable number of flow disks with centres in the area element $\mathrm{d}\boldsymbol{r}$ at position $\boldsymbol{r}$ and velocities within the increment $\mathrm{d}\boldsymbol{c}$ at $\boldsymbol{c}$ is given in terms of the velocity distribution function $f(\boldsymbol{c},\boldsymbol{r},t)$ by $f(\boldsymbol{c},\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{c}\,\mathrm{d}\boldsymbol{r}$. Consequently, the number density n of flow disks is

$$n(\boldsymbol{r},t) = \int f(\boldsymbol{c},\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{c}, \tag{5}$$

where the integration is to be taken over all values of $\boldsymbol{c}$. The density ρ of the flow is mn and the area fraction ν occupied by the disks is $\frac{1}{4}n\pi\sigma^2$. The mean velocity $\boldsymbol{u}$ of the flow is

$$\boldsymbol{u}(\boldsymbol{r},t) \equiv \frac{1}{n}\int \boldsymbol{c}f(\boldsymbol{c},\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{c}, \tag{6}$$

and the granular temperature T is defined by

$$T(\boldsymbol{r},t) \equiv \frac{1}{n}\int \tfrac{1}{2}C^2 f(\boldsymbol{c},\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{c}, \tag{7}$$

where $C \equiv c - u$ is the fluctuation velocity. Knowledge of n, u, T, and all higher moments of f with C determines f.

During a time interval dt, a flow disk with velocity c in dc will collide with a wall disk at some point within the element of angle dk about k provided that: it is moving towards the wall, $g \cdot k > 0$; neighbouring wall disks do not interfere, $-\theta \leqslant k \leqslant \theta$; and its centre is within the parallelogram of area $\bar{\sigma}(g \cdot k)\, dk\, dt$, where $\bar{\sigma} = \frac{1}{2}(d+\sigma)$. Consequently, if the disks were dilute, the number of such collisions within a time dt would be

$$f(c, r+\bar{\sigma}k, t)\,\bar{\sigma}(g \cdot k)\, dk\, dc\, dt, \tag{8}$$

where r is the position of the centre of the wall disk.

Here, however, we wish to consider rather dense systems, so we follow Enskog and introduce into this expression a factor χ that accounts for the effects of excluded area and collisional shielding on the frequency of collisions. Then the probable rate at which such collisions occur is

$$\chi f(c, r+\bar{\sigma}k)\,\bar{\sigma}(g \cdot k)\, dk\, dc. \tag{9}$$

The corresponding collisional frequency per unit length of the wall is obtained by multiplying this by the number α of wall disks per unit length.

Let $\psi = \psi(c)$ be a property associated with a flow disk and write $\Delta\psi \equiv \psi(c') - \psi(c)$ for its change due to a collision with a wall disk. Then, per unit length of the wall, $\mathcal{C}(\psi)$, the rate of change of $\psi(c)$ in collisions, is given by

$$\mathcal{C}(\psi) = \alpha\chi \iint \Delta\psi f(c, r+\bar{\sigma}k)\,\bar{\sigma}(g \cdot k)\, dk\, dc, \tag{10}$$

where the integrations are to be taken over all angles k, $-\theta \leqslant k \leqslant 0$, and velocities c for which a collision is impending, $g \cdot k \geqslant 0$.

When $\psi = mc$, so that $\Delta\psi \equiv m(c'-c) = m(1+e_w)(g \cdot k)\,k$ by (3), we have the collisional rate of supply M of momentum to the flow per unit length of the wall:

$$M = \alpha\chi m(1+e_w) \iint k f(c, r+\bar{\sigma}k)\,\bar{\sigma}(g \cdot k)^2\, dk\, dc, \tag{11}$$

integrated over all possible collisions.

When $\psi = \frac{1}{2}mc^2$, so that $\Delta\psi \equiv \frac{1}{2}m(c'^2 - c^2) = m(1+e_w)(g \cdot k)(U \cdot k) - \frac{1}{2}m(1-e_w^2)(g \cdot k)^2$ by (4), we have the collisional rate of supply E of energy to the flow per unit length of the wall:

$$E = M \cdot U - D, \tag{12}$$

where D is the energy lost to the flow in collisions,

$$D \equiv \tfrac{1}{2}\alpha\chi m(1-e_w^2) \iint f(c, r+\bar{\sigma}k)\,\bar{\sigma}(g \cdot k)^3\, dk\, dc, \tag{13}$$

integrated over all possible collisions.

In order to calculate M and D we must have a definite expression for the velocity distribution function f of the flow disks. Jenkins & Richman (1985*b*), for example, determine the form of f for identical, rough, nearly elastic, circular disks by solving, in an approximate way, the equations governing the evolution of certain higher moments of f. Here, however, we operate in a somewhat cruder fashion and simply suppose that f is Maxwellian:

$$f(c, r, t) = \frac{n}{2\pi T} \exp\left(-\frac{C^2}{2T}\right). \tag{14}$$

This will allow us to describe the essential features of the boundary's influence upon the flow, at least when the collisions between the disks are nearly elastic. We could improve upon the resulting theory, at the expense of more elaborate calculations, by employing the specialization of the distribution function determined by Jenkins & Richman (1985*b*) to dense systems of smooth disks.

When using the distribution function (14) to calculate the rate at which momentum and energy are supplied to the flow by the boundary, we do not assume that the mean flow velocity $\boldsymbol{u}$ at the boundary is equal to the boundary velocity $\boldsymbol{U}$. We suppose that slip occurs, and introduce the slip velocity $\boldsymbol{v}$:

$$\boldsymbol{v} \equiv \boldsymbol{U} - \boldsymbol{u}. \tag{15}$$

Because the boundary is impenetrable, $\boldsymbol{v} \cdot \boldsymbol{n} = 0$. Then, at the wall,

$$f(\boldsymbol{c}, \boldsymbol{r}) = \frac{n}{2\pi T} \exp\left[-\frac{g^2 - 2\boldsymbol{g}\cdot\boldsymbol{v} + v^2}{2T}\right], \tag{16}$$

where $n, \boldsymbol{u}$, and T are evaluated at $\boldsymbol{r}$, the centre of a wall disk.

The expressions for $\boldsymbol{M}$ and D involve $f(\boldsymbol{c}, \boldsymbol{r}+\bar{\sigma}\boldsymbol{k})$, so we first expand this distribution function in a Taylor series about $\boldsymbol{r}$. To truncate the expansion, we introduce a characteristic length L over which the mean fields are supposed to vary, write $\epsilon \equiv \bar{\sigma} L$, and assume that ϵ is small. Then, upon supposing that $\bar{\sigma}\nabla \boldsymbol{u}/T^{\frac{1}{2}}$ is of order $\epsilon^{\frac{1}{2}}$ and that $\bar{\sigma}\nabla T/T$ and $\bar{\sigma}\nabla n/n$ are of order ϵ, we have

$$f(\boldsymbol{c}, \boldsymbol{r}+\bar{\sigma}\boldsymbol{k}) = \left\{1 + \frac{\bar{\sigma}}{T}[(\boldsymbol{k}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{C}\right\} f(\boldsymbol{c}, \boldsymbol{r}), \tag{17}$$

up to an error of order ϵ. Next, we assume that $\bar{\sigma}\boldsymbol{v}/T^{\frac{1}{2}}$ is of order $\epsilon^{\frac{1}{2}}$, expand $f(\boldsymbol{c}, \boldsymbol{r})$, and ignore terms of order ϵ:

$$f(\boldsymbol{c}, \boldsymbol{r}+\bar{\sigma}\boldsymbol{k}) = \frac{n}{2\pi T}\left\{1 - \frac{\bar{\sigma}}{T}[(\boldsymbol{k}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{g} + \frac{\boldsymbol{g}\cdot\boldsymbol{v}}{T}\right\} \exp\left(-\frac{g^2}{2T}\right), \tag{18}$$

where the mean fields are evaluated at $\boldsymbol{r}$.

We use this expansion in the expression (11) for $\boldsymbol{M}$ and, with the help of the integrals evaluated in Appendix B, carry out the integrations. The result, expressed in terms of rectangular Cartesian coordinates, is, up to an error of order ϵ,

$$M_\alpha = \tfrac{1}{2}\rho\chi(1+e_{\mathrm{w}})\,T\left[N_\alpha + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{v_\alpha}{T^{\frac{1}{2}}}(\theta \operatorname{cosec}\theta - \cos\theta) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\bar{\sigma}}{T^{\frac{1}{2}}} u_{\gamma,\beta} I_{\alpha\beta\gamma}\right], \tag{19}$$

where we have used $\bar{\sigma}\alpha = \frac{1}{2}\sin\theta$, and

$$I_{\alpha\beta\gamma} \equiv (\tfrac{8}{3}\sin^2\theta - 2)\,N_\alpha N_\beta N_\gamma - \tfrac{4}{3}\sin^2\theta(N_\alpha \tau_\beta \tau_\gamma + N_\beta \tau_\alpha \tau_\gamma + N_\gamma \tau_\alpha \tau_\beta), \tag{20}$$

in which $\tau_1 = -N_2$ and $\tau_2 = N_1$ are the components of a unit vector tangent to the wall.

In order to obtain a similar expression for the loss of energy D, we must first characterize the size of $(1-e_{\mathrm{w}})$. Here we assume that it is of order ϵ, then so also is the lowest-order contribution to D. In addition, $(1+e_{\mathrm{w}})$ may be replaced by 2 wherever it appears. Upon carrying out the integrations, we obtain

$$D = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \rho\chi(1-e_{\mathrm{w}})\,T^{\frac{3}{2}}\theta \operatorname{cosec}\theta, \tag{21}$$

up to an error of order $\epsilon^{\frac{3}{2}}$.

We later show that the assumptions made regarding the order of various quantities and the retention of different order terms in the lowest-order expressions for $\boldsymbol{M}$ and D lead to a self-consistent solution to a boundary-value problem for the steady shearing of nearly elastic disks. The assumptions correspond to those made by Jenkins & Richman (1985*b*) in their calculation of constitutive relations for the flow.

5. Flow equations and boundary conditions

Jenkins & Richman (1985*b*) derived balance laws for mass, momentum, fluctuation energy, and certain higher moments of the fluctuation velocity for a granular material consisting of identical, rough, inelastic, circular disks. Here we suppose that the flow disks are smooth and that their velocity distribution is Maxwellian. In this event, we require only the balance laws for mass, linear momentum, and fluctuation energy. These have the forms:

$$\dot{\rho}+\rho\nabla\cdot\boldsymbol{u}=0. \tag{22}$$

where the dot indicates the time derivative calculated with respect to the mean motion:

$$\rho\dot{\boldsymbol{u}}=-\nabla\cdot\boldsymbol{P}+n\boldsymbol{F}, \tag{23}$$

where $\boldsymbol{P}$ is the symmetric pressure tensor and $\boldsymbol{F}$ is the external force on a disk; and

$$\rho\dot{T}=-\nabla\cdot\boldsymbol{Q}-\operatorname{tr}(\boldsymbol{P}\cdot\nabla\boldsymbol{u})-\gamma, \tag{24}$$

where $\boldsymbol{Q}$ is the energy flux, tr denotes the trace, and γ is the rate per unit area at which energy is dissipated in collisions.

In general, the fluxes of momentum and energy are due both to transport between collisions and transfer in collisions. Here, however, we shall focus on the collisional contributions and ignore any transport. When the area fraction of flow disks is relatively large, this does not introduce a significant error. We also suppose that the diameter of the flow disks is of the same order as that of the wall disks and that so little energy is lost in collisions between flow disks that their coefficient of restitution e, like e_w, differs from one by a quantity of order ϵ.

The dissipation rate γ is given, up to a sign, by the specialization of equation (117) of Jenkins and Richman (1985*b*) to smooth disks:

$$\gamma=\frac{4(1-e)\,\kappa T}{\sigma^2}. \tag{25}$$

In this,

$$\kappa\equiv\frac{2\rho\sigma\nu g_0\,T^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}, \tag{26}$$

where the function $g_0=g_0(\nu)$, with

$$g_0(\nu)\equiv\frac{16-7\nu}{16(1-\nu)^2}, \tag{27}$$

accounts for the excluded area and the collisional shielding of flow disks. Likewise, the collisional contributions to the energy flux and the pressure tensor may be obtained from equations (100) and (98) of Jenkins & Richman (1985*b*) by ignoring the perturbations to the distribution function and restricting attention to smooth disks. The resulting expressions are

$$\boldsymbol{Q}=-\kappa\nabla T, \tag{28}$$

and

$$\boldsymbol{P}=[2\rho\nu g_0\,T-\tfrac{1}{2}\kappa\operatorname{tr}(\boldsymbol{D})]\,\boldsymbol{1}-\kappa\boldsymbol{D}, \tag{29}$$

where $\boldsymbol{D}$ is the symmetric part of the velocity gradients $\nabla\boldsymbol{u}$ and $\boldsymbol{1}$ is the unit tensor. The constitutive relations (28) and (29) could be improved upon by retaining those terms in the perturbations that contribute at relatively large values of ν.

Boundary conditions are obtained by separately balancing the rate of change of linear momentum and total energy in a rectangle fixed in space with parallel sides of unit length situated at the wall and in the flow. At an impenetrable boundary, as the height of the rectangle goes to zero, the rate M_α at which momentum is supplied to it by the wall and the rate $-P_{\alpha\beta} N_\beta$ at which momentum is supplied to it by the flow must sum to zero. Hence, we have

$$M_\alpha = P_{\alpha\beta} N_\beta. \tag{30}$$

In the same limit, the rate E at which energy is supplied to the rectangle by the wall and the rate $-u_\alpha P_{\alpha\beta} N_\beta - Q_\alpha N_\alpha$ at which energy is supplied to it by the flow must sum to zero. So, upon employing the decomposition (12), the definition (15) of slip velocity, and the boundary condition (30), we obtain

$$M_\alpha v_\alpha - D = Q_\alpha N_\alpha. \tag{31}$$

We note that when there is a slip velocity, the boundary may supply energy to the flow.

We next employ the balance laws (22)–(24), the constitutive relations (25)–(29), and the boundary conditions (30) and (31) to determine the mean fields $\rho, \boldsymbol{u}$, and T and the slip velocity $\boldsymbol{v}$ in a simple, steady flow.

6. A boundary-value problem

We consider a steady rectilinear flow maintained in the absence of gravity by the relative motion of parallel bumpy boundaries. This boundary-value problem has been considered previously by Jenkins & Savage (1983), Haff (1983), and Hui *et al.* (1984) for other, simpler, boundary conditions.

We adopt rectangular Cartesian coordinates x and y and suppose that the walls to which the half-disks are attached are located at $y = \pm\frac{1}{2}L$. The upper wall moves in the x-direction with constant speed U, the lower wall moves with the same speed in the opposite direction. In this steady rectilinear flow the x-component u of the mean velocity, the granular temperature T, and the area fraction ν are functions of y alone.

In this event, the balance of mass (22) is satisfied identically and the x- and y-components of the linear momentum balance (23) reduce to

$$P'_{xy} = 0, \quad P'_{yy} = 0. \tag{32a, b}$$

where a prime denotes a derivative with respect to y. In these, from (29), we have

$$P_{xy} = -\tfrac{1}{2}\kappa u', \quad P_{yy} = 2\rho\nu g_0 T. \tag{33a, b}$$

Equations (32) may be integrated immediately to

$$\tfrac{1}{2}\kappa u' = S, \quad 2\rho\nu g_0 T = N, \tag{34a, b}$$

where S and N are, respectively, the constant values of the shear stress and pressure. We may use (34b) with the definition (26) of κ to write κ in terms of N:

$$\kappa = \frac{N\sigma}{(\pi T)^{\frac{1}{2}}}; \tag{35}$$

then (34a), becomes

$$u' = \frac{2(\pi T)^{\frac{1}{2}} S}{N\sigma}. \tag{36}$$

We note that when $\sigma u'/T^{\frac{1}{2}}$ is of order $\epsilon^{\frac{1}{2}}$, then so is S/N.

In this flow the energy balance (24) reduces to

$$Q'_y + P_{xy} u' + \gamma = 0, \tag{37}$$

where P_{xy} is given by (33*a*), and, from (28) and (25),

$$Q_y = -\kappa T', \quad \gamma = \frac{4(1-e)\kappa T}{\sigma^2}. \tag{38a, b}$$

Upon employing these and using (35) and (36) to eliminate, respectively, κ and u' from the resulting equation, we obtain a linear second-order differential equation for $w \equiv T^{\frac{1}{2}}$:

$$L^2 w'' - \lambda^2 w = 0, \tag{39}$$

where

$$\lambda^2 \equiv \left[2(1-e) - \pi\left(\frac{S}{N}\right)^2\right]\left(\frac{L}{\sigma}\right)^2. \tag{40}$$

The solution of (39) that satisfies the condition $w'(0) = 0$ is

$$w = A\cosh\left(\frac{\lambda y}{L}\right), \tag{41}$$

where A is a constant to be determined. Then the corresponding solution of (36) that satisfies the condition $u(0) = 0$ is

$$u = 2\pi^{\frac{1}{2}}\frac{S}{N}\frac{L}{\sigma}\frac{A}{\lambda}\sinh\left(\frac{\lambda y}{L}\right). \tag{42}$$

At $y = \frac{1}{2}L$, u differs from U by the slip velocity v, so

$$A = \frac{1}{2\pi^{\frac{1}{2}}}\frac{\lambda(U-v)}{\sinh(\frac{1}{2}\lambda)}\frac{N}{S}\frac{\sigma}{L}. \tag{43}$$

Then

$$w = \frac{1}{2\pi^{\frac{1}{2}}}\frac{\lambda(U-v)}{\sinh(\frac{1}{2}\lambda)}\frac{N}{S}\frac{\sigma}{L}\cosh\left(\frac{\lambda y}{L}\right) \tag{44}$$

and

$$u = \frac{(U-v)}{\sinh(\frac{1}{2}\lambda)}\sinh\left(\frac{\lambda y}{L}\right). \tag{45}$$

We next use the boundary conditions (30) and (31) to determine v, λ and $\nu(\frac{1}{2}L)$. Because S/N is given in terms of λ through (40), the solutions for w and u will then be complete. The variation of ν is governed by (34*b*) in which N is determined by the knowledge of ν and w at $\frac{1}{2}L$.

At the upper wall, $N_x = \tau_y = 0$, $N_y = -\tau_x = -1$ and the non-vanishing components of $I_{\alpha\beta\gamma}$ are

$$I_{yyy} = 2 - \tfrac{2}{3}\sin^2\theta \tag{46}$$

and

$$I_{yxx} = I_{xyx} = I_{xxy} = \tfrac{2}{3}\sin^2\theta. \tag{47}$$

The normal component of the boundary condition (30) is

$$N_\alpha M_\alpha = N_\alpha P_{\alpha\beta} N_\beta. \tag{48}$$

With (19), (33*b*), and (34*b*), this becomes

$$\rho\chi T = N \quad \text{or} \quad \chi = 2\nu g_0. \tag{49a, b}$$

Consequently, if the dependence of χ on ν could be determined, then (49*b*) fixes the value of ν at the wall. We anticipate that χ also depends upon the size and spacing of the wall disks; for, although the shielding of a wall disk by its neighbours has been accounted for explicitly, χ should at least include the effect of flow disks shielding a wall disk. If, for example, we were to ignore this and identify χ with g_0, then $\nu(\frac{1}{2}L) = \frac{1}{2}$, independent of the geometry of the boundary.

The tangential component of (30) is

$$\tau_\alpha M_\alpha = \tau_\alpha P_{\alpha\beta} N_\beta, \tag{50}$$

or

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \rho\chi T\left[\frac{v}{w}(\theta \operatorname{cosec}\theta - \cos\theta) + \frac{2}{3}\frac{\bar{\sigma}u'}{w}\sin^2\theta\right] = \tfrac{1}{2}\kappa u'. \tag{51}$$

If, in this, we use (49a) and the first integrals (34a) and (36), it becomes

$$\frac{v}{w} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{S}{N} \frac{[1-(4\sqrt{2}\bar{\sigma}/3\sigma)\sin^2\theta]}{(\theta \operatorname{cosec}\theta - \cos\theta)}. \tag{52}$$

This is a balance between quantities of order $\epsilon^{\frac{1}{2}}$. We note that when $(4\sqrt{2}\bar{\sigma}/3\sigma)\sin^2\theta$ is greater than one, the slip velocity is negative, which appears to be unrealistic. However the flow field has been somewhat artificially extended to include the entire region between the tips of the wall disks and the wall. It is possible to show that this expression for the slip velocity leads to mean flow velocities at the tips of the wall disks that can never be greater than the wall velocity.

The energy boundary condition (31) at the upper wall may be written with the help of (19)–(21), (38), (35), and (49b) as

$$\left(\frac{v}{w}\right)^2 (\theta \operatorname{cosec}\theta - \cos\theta) + \frac{v}{w}\left(\frac{2}{3}\frac{\bar{\sigma}u'}{w}\sin^2\theta\right) - \left[\sqrt{2}\frac{\sigma w'}{w} + (1-e_{\mathrm{w}})\,\theta \operatorname{cosec}\theta\right] = 0. \tag{53}$$

When v/w and $\bar{\sigma}u'/w$ are of order $\epsilon^{\frac{1}{2}}$ and $\bar{\sigma}w'/w$ and $(1-e_{\mathrm{w}})$ are of order ϵ, the energy boundary condition is a relation between quantities of order ϵ. In this we use (52) and (36) to express v/w and $\bar{\sigma}u'/w$ in terms of S/N and replace $\sigma w'/w$ by its value at the wall calculated from the solution (44) and obtain

$$2\sqrt{2}\frac{\sigma}{L}\tfrac{1}{2}\lambda \tanh(\tfrac{1}{2}\lambda) = \tfrac{1}{2}\pi\left(\frac{S}{N}\right)^2 \frac{[1-(4\sqrt{2}\bar{\sigma}/3\sigma)\sin^2\theta]}{(\theta \operatorname{cosec}\theta - \cos\theta)} - (1-e_{\mathrm{w}})\,\theta \operatorname{cosec}\theta. \tag{54}$$

Now, from the definition (40) of λ,

$$\tfrac{1}{2}\pi\left(\frac{S}{N}\right)^2 = (1-e) - 2\left(\frac{\sigma}{L}\right)^2 (\tfrac{1}{2}\lambda)^2; \tag{55}$$

so (54) is a transcendental equation to be solved for $\tfrac{1}{2}\lambda$.

The solution of this equation may be simplified by the following order of magnitude considerations. We have shown that the assumption that $\bar{\sigma}u'/w$ is of order $\epsilon^{\frac{1}{2}}$ implies that S/N is also of order $\epsilon^{\frac{1}{2}}$. We have also assumed that $\bar{\sigma}w'/w$ is of order ϵ and, using (44), this implies that $(\bar{\sigma}/L)\,\lambda \tanh(\tfrac{1}{2}\lambda)$ is of order ϵ. Consequently, λ is of order one. Then, with $(1-e)$ of order ϵ, (55) gives, up to an error of order ϵ^2,

$$\left(\frac{S}{N}\right)^2 = \frac{2(1-e)}{\pi}. \tag{56}$$

With this, (54) simplifies to

$$\tfrac{1}{2}\lambda \tanh(\tfrac{1}{2}\lambda) = \frac{(1-e_{\mathrm{w}})}{2\sqrt{2}}\frac{L}{\sigma}\frac{\theta}{\sin\theta}\left\{\frac{(1-e)}{(1-e_{\mathrm{w}})}\frac{\sin\theta\,[1-(4\sqrt{2}\bar{\sigma}/3\sigma)\sin^2\theta]}{\theta(\theta \operatorname{cosec}\theta - \cos\theta)} - 1\right\}. \tag{57}$$

When the quantity in brackets is positive, then λ is real and the solutions (44) and (45) involving the hyperbolic functions apply. In this case, the granular temperature attains its maximum value at the wall and decreases towards the centre, and the wall supplies fluctuation energy to the flow. Then equation (34b) shows that the area fraction has its greatest value at the centreline and decreases towards the walls.

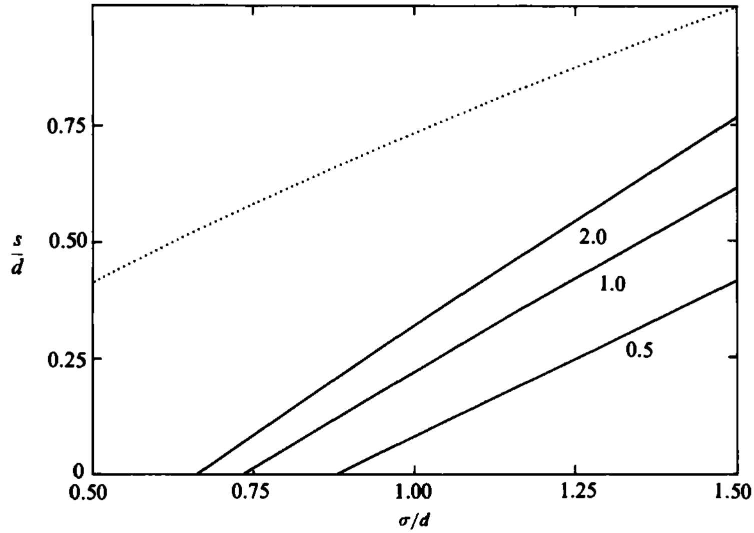

FIGURE 2. Curves of $\lambda = 0$ in the space of diameter ratio σ/d and non-dimensional spacing s/d for $(1-e)/(1-e_w) = \frac{1}{2}$, 1, and 2. The dotted curve is the graph of $s/d = -1+(1+2\sigma/d)^{\frac{1}{2}}$, above which flow disks collide with the flat wall.

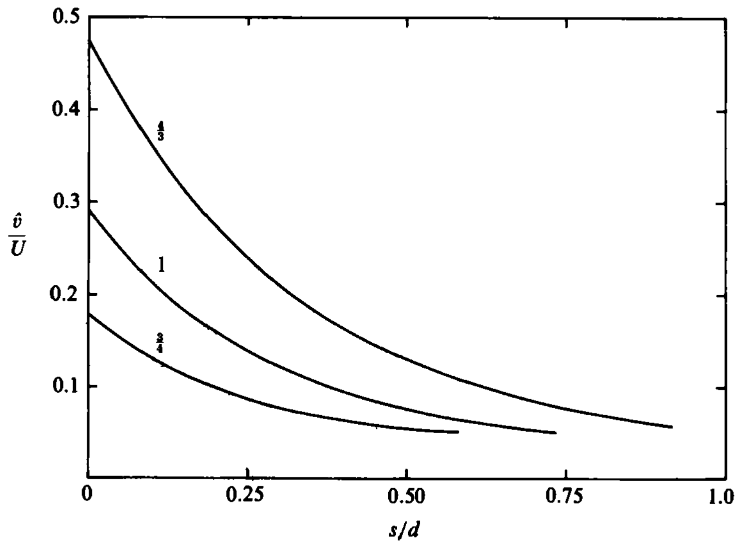

FIGURE 3. The non-dimensional slip velocity $\hat{v}/U$ at the tips of the wall disks, versus the non-dimensional spacing s/d for diameter ratios $\sigma/d = \frac{4}{3}$, 1 and $\frac{3}{4}$ when $\sigma/L = \frac{1}{11}$ and $e = e_w = 0.9$.

When the quantity in brackets is negative, then λ is imaginary and the solutions corresponding to (44) and (45) may be written in terms of trigonometric functions. The determinations of λ leading to solutions involving negative values of the granular temperature must be excluded. In this case, the granular temperature is greatest on the centreline and decreases toward the walls where fluctuation energy is absorbed from the flow. The area fraction is greatest at the walls and decreases toward the centreline.

The critical case occurs when the quantity in brackets, and λ, vanish. This is a simple homogeneous shearing flow with T, u' and ν constant. In figure 2 we show the locus of $\lambda = 0$ in the space of diameter ratio σ/d and non-dimensional spacing s/d

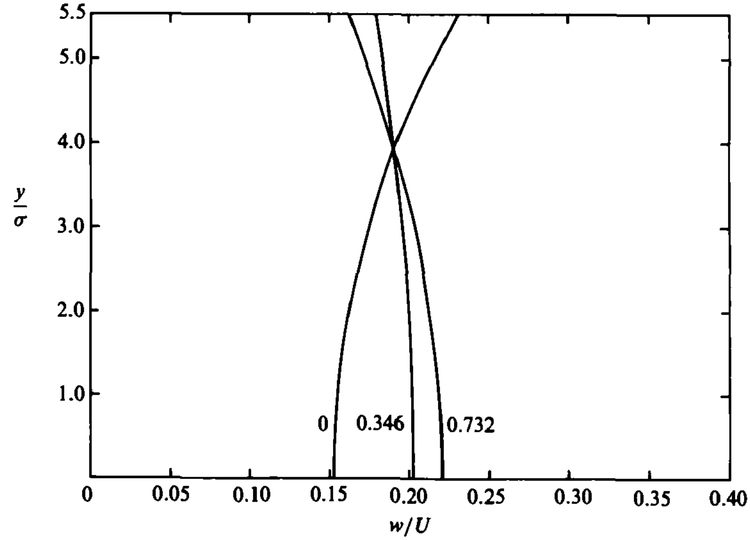

FIGURE 4. Distributions across the gap of the measure w/U of the granular temperature when $\sigma = d$, $e = e_w = 0.9$, and $\sigma/L = \frac{1}{11}$ for $s/d = 0, 0.346$ and 0.732.

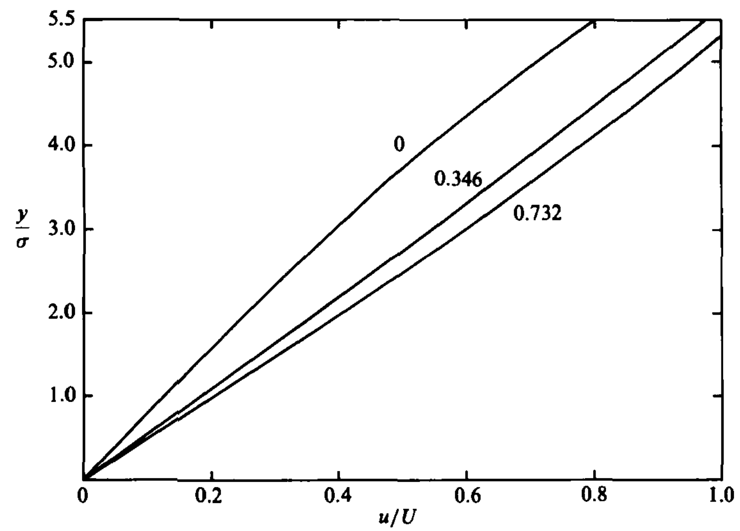

FIGURE 5. Distributions of the non-dimensional mean velocity u/U across the gap when $\sigma = d$, $e = e_w = 0.9$ and $\sigma/L = \frac{1}{11}$ for $s/d = 0, 0.346$ and 0.732.

for several values of $(1-e)/(1-e_w)$. In the region of the space above a curve $\lambda = 0$, the solutions are given in terms of trigonometric functions; below such a curve, in terms of hyperbolic functions.

With λ determined, the slip velocity is obtained by employing (44) in (52):

$$\frac{U}{v} = 1 + \frac{2\sqrt{2}}{\lambda}\frac{L}{\sigma}\frac{(\theta \operatorname{cosec}\theta - \cos\theta)}{[1-(4\sqrt{2}\bar{\sigma}/3\sigma)\sin^2\theta]}\tanh(\tfrac{1}{2}\lambda). \tag{58}$$

With this, the temperature and velocity profiles are given in terms of known quantities by (44) and (45).

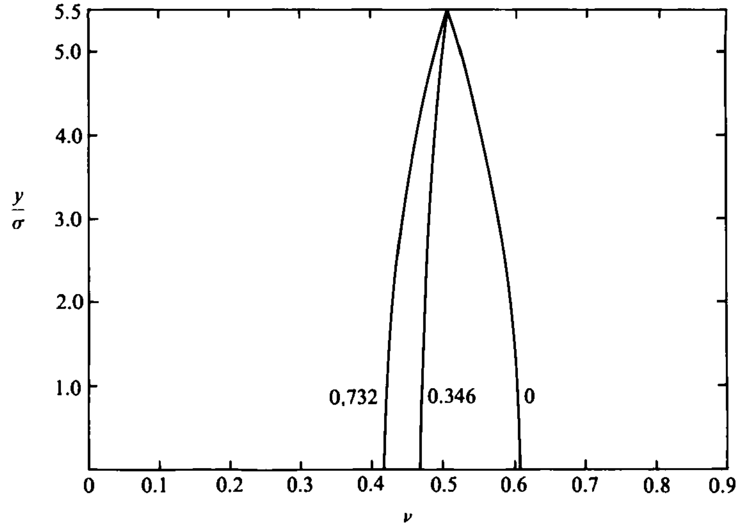

FIGURE 6. Distribution of area fraction ν across the gap when $\sigma = d$, $e = e_w = 0.9$ and $\sigma/L = \frac{1}{11}$ for $s/d = 0.346$ and 0.732.

We have obtained solutions for several values of the diameter ratio σ/d, assuming that $e = e_w = 0.9$, and $\sigma/L = \frac{1}{11}$. In figure 3 we indicate how the slip velocity $\hat{v}$ at the tips of the disks varies with wall-disk spacing for these diameter ratios. In figures 4 and 5 we show the profiles of granular temperature and mean velocity for wall disks and flow disks of equal diameter. In each figure profiles are given for the two extremes of wall-disk spacing, $s/d = 0$ and 0.732, and for the spacing $s/d = 0.346$. To this point the analysis is independent of the form of χ.

For such solutions, knowledge of χ fixes ν at the wall. Then N is determined from (34*b*), and the variation in area fraction is obtained from the relation

$$\nu^2 g_0(\nu) = \frac{\pi \sigma^2 N}{8m[w(y)]^2} \tag{59}$$

by solving numerically for ν at various values of y. For example, upon supposing that $\chi \equiv g_0$, we obtain the three area-fraction profiles shown in figure 6. When $s/d = 0.346$ the spacing between wall disks is the same as the mean separation between the flow disks at $\nu = \frac{1}{2}$.

7. Discussion

We have taken care to indicate the assumed orders of magnitude of quantities that enter into the calculations and to show that these assumptions lead to relations between quantities of the same orders of magnitude. This was done because energy is balanced at a higher order than momentum and, unless such care is taken, terms may be incorrectly included or excluded from the energy balances. Of course these assumptions restrict the range of applicability of the theory to wall disks and flow disks that are smooth, nearly elastic, and whose diameters do not differ greatly. Still, the results should be of interest to those carrying out experiments and numerical simulations.

For example, our analysis shows that a steady shearing flow generated by the relative motion of parallel plates a fixed distance apart will only be obtained for a definite number of flow disks across the gap. If an experiment or simulation is initiated with the wrong number of disks, steady flow will not be attained or the disks may arrange themselves into rigidly translating layers to create internal boundaries. Also, if a homogeneous shearing flow is desired, it can only be achieved by tuning the boundary to the flow. For smooth, nearly elastic disks, figure 2 indicates how this can be done for several values of $(1-e)/(1-e_w)$.

The results displayed in figure 3 show that the slip velocity at the tips of the disks may be decreased by either increasing the spacing of the wall disks while holding the ratio of the diameters fixed or by decreasing the diameter of the flow disks relative to the wall disks at fixed spacing. Either change results in a rougher wall.

The distributions of w/U, u/U and ν across the gap shown in figures 4, 5 and 6 exhibit features that we have already described. As can be determined from figure 2 for the case $e = e_w$ and $\sigma = d$, both hyperbolic and trigonometric solutions are obtained, depending on the spacing of the wall disks. Because of our identification of the correction to the collision frequency at the wall with that in the flow, the area fraction at the wall for each spacing of the wall disks is equal to $\frac{1}{2}$. However at the tips of the wall disks the area fractions are different for each spacing.

The extension of these results to systems involving significant dissipation of energy, such as rough, inelastic disks, awaits the development of constitutive relations for these materials. It is likely that these will involve measures of the internal state of the material that are more elaborate than the granular temperature and will be nonlinear in, at least, the gradients of the mean velocity.

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Appendix A

We consider the flow of a granular material consisting of identical, smooth, nearly elastic, spheres of mass m and diameter σ and calculate the boundary conditions for the pressure tensor and energy flux vector at a bumpy boundary. The boundary is composed of identical, smooth, nearly elastic, hemispheres of diameter d attached to a flat wall.

The hemispheres are assumed to be randomly distributed over the wall so that the mean spacing between their edges is s. Then, in mean, the neighbourhood of any hemisphere is symmetric about an axis through its centre and normal to the wall. Equivalently, the nearest neighbour of any hemisphere is, on average, half of a torus with inner diameter $d+2s$, outer diameter $3d+2s$, and height $\frac{1}{2}d$. A plane through the centre of the hemisphere and its mean neighbouring torus is shown in figure 1.

Clearly with this simplified characterization of the boundary geometry the condition prohibiting collisions with the flat wall is identical with that for disks, and the angle θ retains its significance and is defined as before. The average number α of hemispheres per unit area of the flat wall is $4/\pi(d+s)^2$. The extension of the notation from two to three dimensions is, in most cases, obvious; we supply redefinitions when necessary.

The expression corresponding to (10) for the collisional rate of production of $\psi(\boldsymbol{c})$, per unit area of the flat wall, is

$$C(\psi) = \alpha\chi \int\int \Delta\psi f(\boldsymbol{c}, \boldsymbol{r}+\bar{\sigma}\boldsymbol{k})\,\bar{\sigma}^2(\boldsymbol{g}\cdot\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}, \tag{A 1}$$

where $\boldsymbol{k}$ is the unit vector directed from the centre of the hemisphere to the centre of a colliding sphere; $\mathrm{d}\boldsymbol{k}$ is the element of solid angle about $\boldsymbol{k}$; and the integrations are to be carried out over those values of $\boldsymbol{c}$, polar angle k, and circumferential angle ϕ for which a collision can occur, $0 \leqslant k \leqslant \theta$ and $\boldsymbol{g}\cdot\boldsymbol{k} > 0$.

We employ the three-dimensional Maxwellian

$$f(\boldsymbol{c},\boldsymbol{r},t) = \left(\frac{n}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{C^2}{2T}\right), \tag{A 2}$$

where

$$\tfrac{3}{2}T \equiv \frac{1}{n}\iint \tfrac{1}{2}C^2 f(\boldsymbol{c},\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{c}, \tag{A 3}$$

and the expansion corresponding to (18). Then, with $\psi = m\boldsymbol{c}$ in (A 1), we integrate as outlined in Appendix B and obtain

$$M_i = \tfrac{1}{2}\rho\chi(1+e_\mathrm{w})\,T\left\{N_i - \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{v_i}{T^{\frac{1}{2}}}[\cos\theta + 2\operatorname{cosec}^2\theta(1-\cos\theta)] + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\bar{\sigma}}{T^{\frac{1}{2}}}u_{k,j}\,I_{ijk}\right\}, \tag{A 4}$$

where we have used $\bar{\sigma}^2\alpha = 1/\pi\sin^2\theta$, and

$$I_{ijk} \equiv (\sin^2\theta - 2)\,N_i N_j N_k - \tfrac{1}{2}\sin^2\theta[N_i(\tau_j\tau_k + t_j t_k) + N_j(\tau_i\tau_k + t_i t_k) + N_k(\tau_i\tau_j + t_i t_j)]. \tag{A 5}$$

in which $\boldsymbol{N}, \boldsymbol{\tau}, \boldsymbol{t}$ is an orthonormal triad. In a similar fashion, with $\psi = \frac{1}{2}mc^2$ in (A 1), we obtain

$$E = \boldsymbol{M}\cdot\boldsymbol{U} - D \tag{A 6}$$

with

$$D \equiv \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\rho\chi(1-e_\mathrm{w}^2)\,T^{\frac{3}{2}}(1-\cos\theta)\operatorname{cosec}^2\theta. \tag{A 7}$$

The boundary conditions corresponding to (19) and (21) are, then,

$$M_i = P_{ik}N_k \tag{A 8}$$

and

$$M_i v_i - D = Q_i N_k, \tag{A 9}$$

with $\boldsymbol{M}$ and D given by (A 4) and (A 7) respectively.

Appendix B

Here we outline the details of the integrations involved in the calculation of $\boldsymbol{M}$ and D for disks and spheres.

For disks, the typical integral is of the form

$$\iint \boldsymbol{g}\ldots\boldsymbol{g}\boldsymbol{k}\ldots\boldsymbol{k}(\boldsymbol{g}\cdot\boldsymbol{k})^p \exp\left(-\frac{g^2}{a}\right)\mathrm{d}k\,\mathrm{d}\boldsymbol{g} \tag{B 1}$$

where the integration is over all $\boldsymbol{g}\cdot\boldsymbol{k} > 0$ with $-\theta \leqslant k \leqslant \theta$, p is an integer, and a is a constant.

We write $\boldsymbol{g} = G\boldsymbol{k} + H\boldsymbol{j}$, where $\boldsymbol{j}$ is a unit vector normal to $\boldsymbol{k}$ in the plane; express the tensor product of the vectors $\boldsymbol{g}$ in terms of these components; and carry out the integrations over G, $0 < G < \infty$, and H, $-\infty < H < \infty$, with the help of the standard results

$$\int_0^\infty s^q \exp\left(-\frac{s^2}{a}\right)\mathrm{d}s = \tfrac{1}{2}\Gamma\left(\frac{q+1}{2}\right)a^{(q+1)/2}, \tag{B 2}$$

when q is even, and

$$\int_0^\infty \exp\left(-\frac{s^2}{d}\right)\mathrm{d}s = \frac{1}{2}\left(\frac{q-1}{2}\right)!\,a^{(q+1)/2}, \tag{B 3}$$

when q is odd.

The remaining integrations ar of the form

$$\int \boldsymbol{k}\ldots\boldsymbol{k}\,\mathrm{d}k. \tag{B 4}$$

We write $\boldsymbol{k} = \cos k\boldsymbol{N} + \sin k\boldsymbol{\tau}$, express the tensor product of the vectors $\boldsymbol{k}$ in terms of these components, and integrate over $\boldsymbol{k}$, $-\theta < k < \theta$. The results necessary for the calculation of $\boldsymbol{M}$ and D are

$$\int \mathrm{d}k = 2\theta; \tag{B 5}$$

$$\int k_\alpha\,\mathrm{d}k = 2\sin\theta N_\alpha; \tag{B 6}$$

$$\int k_\alpha k_\beta\,\mathrm{d}k = \tfrac{1}{2}(2\theta + \sin 2\theta)\,N_\alpha N_\beta + \tfrac{1}{2}(2\theta - \sin 2\theta)\,\tau_\alpha \tau_\beta; \tag{B 7}$$

and

$$\int k_\alpha k_\beta k_\gamma\,\mathrm{d}k = -\sin\theta I_{\alpha\beta\gamma}, \tag{B 8}$$

where $I_{\alpha\beta\gamma}$ is defined in (20).

For spheres, the typical integral is of the form (B 1) with $\boldsymbol{g}$ and $\boldsymbol{k}$ vectors in three dimensions and $\mathrm{d}k$ replaced by the element of solid angle $\mathrm{d}\boldsymbol{k}$. When $\boldsymbol{g}$ is written in terms of its components with respect to the orthonormal basis $\boldsymbol{k}, \boldsymbol{j}, \boldsymbol{i}$ and the tensor products of the vectors $\boldsymbol{g}$ are expressed in terms of these components, the integrations over the components are carried out as in the two-dimensional case.

There remain integrals of the form

$$\int \boldsymbol{k}\ldots\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}. \tag{B 9}$$

We write $\boldsymbol{k} = \cos k\boldsymbol{N} + \sin k\cos\phi\boldsymbol{\tau} + \sin k\sin\phi\boldsymbol{t}$ and $\mathrm{d}\boldsymbol{k} = \sin k\,\mathrm{d}k\,\mathrm{d}\phi$; express the tensor product of the vectors $\boldsymbol{k}$ in terms of these components; and integrate over $\phi, 0 \leqslant \phi \leqslant 2\pi$, and $k, 0 \leqslant k \leqslant \theta$. The results necessary for the calculation of $\boldsymbol{M}$ and D are

$$\int \mathrm{d}\boldsymbol{k} = 2\pi(1-\cos\theta); \tag{B 10}$$

$$\int k_i\,\mathrm{d}\boldsymbol{k} = \pi\sin^2\theta N_i; \tag{B 11}$$

$$\int k_i k_j\,\mathrm{d}\boldsymbol{k} = \tfrac{2}{3}\pi(1-\cos^3\theta)\,N_i N_j + \tfrac{1}{3}\pi(2 - 3\cos\theta + \cos^3\theta)\,(\tau_i\tau_j + t_i t_j); \tag{B 12}$$

and

$$\int k_i k_j k_k\,\mathrm{d}\boldsymbol{k} = -\tfrac{1}{2}\pi\sin^2\theta I_{ijk}, \tag{B 13}$$

where I_{ijk} is defined in (A 5).

REFERENCES

Ackermann, N. L. & Shen, H. H. 1983 Rapid shear flow of densely packed granular solids. In *Mechanics of Granular Materials: New Models and Constitutive Relations* (ed. J. T. Jenkins and M. Satake), pp. 295–304. Elsevier.

Bagnold, R. A. 1954 Experiments on a gravity-free dispersion of large solid spheres in a Newtonian fluid under shear. *Proc. R. Soc. Lond.* A **225**, 49–63.

Campbell, C. S. & Brennen, C. E. 1983 Computer simulation of shear flows of granular material.

In *Mechanics of Granular Materials: New Models and Constitutive Relations* (ed. J. T. Jenkins & M. Satake), pp. 313–326. Elsevier.

CAMPBELL, C. S. & BRENNEN, C. E. 1985 Chute flow of granular material: some computer simulations. *Trans. ASME* E: *J. Appl. Mech.* **52**, 172–178.

CAMPBELL, C. S. & GONG, A. 1986 The stress tensor in a two-dimensional granular shear flow. *J. Fluid Mech.* **164**, 107–125.

CHAPMAN, S. & COWLING, T. G. 1970 *The Mathematical Theory of Non-Uniform gases*, 3rd edn. Cambridge University Press.

HAFF, P. K. 1983 Grain flow as a fluid-mechanical phenomenon. *J. Fluid Mech.* **134**, 401–430.

HANES, D. M. & INMAN, D. L. 1985 Observations of rapidly flowing granular-fluid mixtures. *J. Fluid Mech.* **150**, 357–380.

HUI, K., HAFF, P. K., UNGAR, J. E. & JACKSON, R. 1984 Boundary conditions for high-shear grain flows. *J. Fluid Mech.* **145**, 223–233.

JENKINS, J. T. & MANCINI, F. 1986 Balance laws and constitutive relations for plane flows of a dense, binary mixture of smooth, nearly elastic, circular disks. *Trans. ASME* E: *J. Appl. Mech.* (in press).

JENKINS, J. T. & RICHMAN, M. W. 1985*a* Grad's 13-moment system for a dense gas of inelastic spheres. *Arch. Rat. Mech. Anal.* **87**, 355–377.

JENKINS, J. T. & RICHMAN, M. W. 1985*b* Kinetic theory for plane flows of a dense gas of identical, rough, inelastic, circular disks. *Phys. Fluids* **28**, 3485–3494.

JENKINS, J. T. & SAVAGE, S. B. 1983 A theory for the rapid flow of identical, smooth, nearly elastic spherical particles. *J. Fluid Mech.* **130**, 187–202.

LUN, C. K. K. & SAVAGE, S. B. 1986 A simple kinetic theory for granular flow of rough, inelastic, spherical prticles. *Trans. ASME* E: *J. Appl. Mech.* (in press).

LUN, C. K. K., SAVAGE, S. B., JEFFREY, D. J. & CHEPURNIY, N. 1984 Kinetic theories for granular flow: inelastic particles in Couette flow and slightly inelastic particles in a general flow field. *J. Fluid Mech.* **140**, 223–256.

OGAWA, S., UMEMURA, A. & OSHIMA, N. 1980 On the equations of fully fluidized granular materials. *Z. angew. Math. Phys.* **31**, 483–493.

SAVAGE, S. B. 1978 Experiments on shear flows of cohesionless granular materials. In *Continuum Mechanical and Statistical Approaches in the Mechanics of Granular Materials* (ed. S. C. Cowin & M. Satake), pp. 241–254. Gakujutsu Bunken Fukyu-kai.

SAVAGE, S. B. & JEFFREY, D. J. 1981 The stress tensor in a granular flow at high shear rates. *J. Fluid Mech.* **110**, 255–272.

SAVAGE, S. B. & MCKEOWN, S. 1983 Shear stresses developed during rapid shear of dense concentrations of large spherical particles between concentric rotating cylinders. *J. Fluid Mech.* **127**, 453–472.

SAVAGE, S. B. & SAYED, M. 1984 Stresses developed by dry cohesionless granular materials sheared in an annular shear cell. *J. Fluid Mech.* **42**, 391–430.

SHEN, H. H. 1984 Stresses in rapid deformations of spherical solids with two sizes. *J. Particulate Sci. Tech.* **2**, 37–56.

SHEN, H. H. & ACKERMAN, N. L. 1984 Constitutive equations for a simple shear flow of disk-shaped granular mixtures. *Intl J. Engng Sci.* **22**, 829–843.

WALTON, O. R. 1983 Particle dynamics calculations of shear flow. In *Mechanics of Granular Materials: New Models and Constitutive Relations* (ed. J. T. Jenkins & M. Satake), pp. 327–338. Elsevier.